



M337/Practice exam 2

Module Examination

Complex analysis

Time allowed: 3 hours

There are **two parts** to this examination.

In **Part 1** you should **submit answers to all 6 questions**. Each question is worth 10% of the total mark.

In **Part 2** you should **submit answers to 2 out of the 3 questions**. Each question is worth 20% of the total mark.

Do not submit more than two answers for Part 2. If you submit answers to all three Part 2 questions, then only Questions 7 and 8 will be marked.

Include all your working, as some marks are awarded for this.

Write your answers in **pen**, though you may draw diagrams in pencil.

Start your answer to each question on a new page, clearly indicating the number of the question.

Crossed out work will not be marked.

Follow the instructions in the online timed examination for how to submit your work. Further information about completing and submitting your examination work is in the *Instructions and guidance for your remote examination* document on the module website.

Part 1

You should **submit answers to all questions** from Part 1.

Each question is worth 10%.

Question 1

Determine each of the following complex numbers in *Cartesian* form, simplifying your answers as far as possible.

(a) $\exp(2 + i\pi/4)$ [2]

(b) $1 + 2\sinh^2(i\pi/6)$ [2]

(c) $\text{Log}(-1 - i)$ [3]

(d) $(-i)^{-i}$ [3]

Question 2

Evaluate the following integrals with $C = \{z : |z| = 1\}$. Name any standard results that you use and check that their hypotheses are satisfied.

(a) (i) $\int_C \frac{1}{z^6} dz$ [2]

(ii) $\int_C \frac{\sin z}{z} dz$ [2]

(iii) $\int_C \frac{\sin z}{z^6} dz$ [4]

(b) Explain how if at all the answers you obtained in part (a) would change if C was replaced by another circle centred at the origin. [2]

Question 3

Let

$$f(z) = \sum_{n=0}^{\infty} z^{2n} \quad (|z| < 1)$$

and

$$g(z) = -\sum_{n=1}^{\infty} \frac{1}{z^{2n}} \quad (|z| > 1).$$

(a) Explain why f and g are not direct analytic continuations of each other. [1]

(b) Find an analytic function h such that

$$f(z) = h(z), \quad \text{for } |z| < 1,$$

and

$$g(z) = h(z), \quad \text{for } |z| > 1. \quad [6]$$

(c) Deduce that f and g are indirect analytic continuations of each other. [3]

Question 4

Consider the equation

$$z^5 + 3z^3 - 1 = 0.$$

- (a) Use Rouché's Theorem to determine the number of solutions of the equation in the annulus $\{z : 1 < |z| < 2\}$. [8]
- (b) Prove that at most two of the solutions of the equation lie in the upper half-plane $\{z : \operatorname{Im} z > 0\}$. [2]

Question 5

Let q be the velocity function

$$q(z) = \frac{1}{\bar{z} + i}$$

for an ideal flow with flow region $\mathbb{C} - \{i\}$.

- (a) Prove that the function

$$\Psi(z) = \operatorname{Arg}(z - i)$$

is a stream function for this flow, and hence find an equation for the streamline through the point $1 + 2i$. [5]

- (b) Sketch this streamline and indicate the direction of flow. [2]
- (c) Use the Circulation and Flux Contour Integral to classify the point i as a source, sink or vortex of the flow. [3]

Question 6

- (a) Let f be the function

$$f(z) = \frac{1}{2} \left(z + \frac{1}{z} \right).$$

Find the fixed points of f and classify them as attracting, repelling or indifferent, identifying any attracting fixed points that are super-attracting. [4]

- (b) Determine whether or not each of the following points c lies in the Mandelbrot set.
- (i) $c = -1 - i$. [3]
- (ii) $c = -\frac{1}{4}i$ [3]

Part 2

You should **submit answers to two questions** from Part 2. If you submit answers to all three Part 2 questions, then only Questions 7 and 8 will be marked.

Each question is worth 20%.

Question 7

(a) (i) Find an example of a compact set A for which $A - \mathbb{R}$ is not compact. Justify your example briefly. [3]

(ii) Find an example of a region B for which $B - \mathbb{R}$ is not a region. Justify your example briefly. [3]

(iii) Prove that if C is an open set then $C - \mathbb{R}$ is also an open set. [4]

(b) Let f be the function

$$f(z) = z^2 + 2(\operatorname{Im} z)^2 + 4i(\operatorname{Re} z)^2.$$

(i) Prove that

$$f(x + iy) = (x^2 + y^2) + i(4x^2 + 2xy). \quad [1]$$

(ii) Use the Cauchy–Riemann Theorem and its converse to determine all the points $z = x + iy$ of \mathbb{C} at which f is differentiable. [9]

Question 8

(a) Let f be the function

$$f(z) = \frac{\sin z}{z^2(z-2)^2}.$$

Write down the singularities of f and find the residues of f at these singularities. [7]

(b) Let g be the function

$$g(z) = \frac{3}{z(z-3)}.$$

Find the Laurent series about 1 for f on the annulus $\{z : 1 < |z-1| < 2\}$, giving the constant term and two terms on each side of it. [10]

(c) Suppose that entire functions f and g satisfy $f(0) = g(0)$ and

$$|f(z) - g(z)| < 1, \quad \text{for } z \in \mathbb{C}.$$

Prove that f and g are equal. [3]

Question 9

Let

$$\mathcal{R} = \{z : |z - 1 - i| < 1, \operatorname{Re} z + \operatorname{Im} z < 1\},$$

$$\mathcal{S} = \{z : 0 < \operatorname{Arg} z < \pi/4\},$$

$$\mathcal{T} = \{z : 0 < \operatorname{Arg} z < \pi\}.$$

- (a) Sketch the regions \mathcal{R} , \mathcal{S} and \mathcal{T} . [3]
- (b) Find a positive number λ such that the point $\lambda(1 + i)$ lies on the boundary of \mathcal{R} . [1]
- (c) Use the strategy for mapping lunes to determine a one-to-one conformal mapping f from \mathcal{R} onto \mathcal{S} . [7]
- (d) Determine a one-to-one conformal mapping g from \mathcal{S} onto \mathcal{T} . [2]
- (e) Hence determine a one-to-one conformal mapping h from \mathcal{R} onto \mathcal{T} , and find the rule for the inverse function h^{-1} . [5]
- (f) Explain why h^{-1} is a one-to-one conformal mapping from \mathcal{T} onto \mathcal{R} . [2]

[END OF QUESTION PAPER]